DOCUMENT RESUME

ED 167 385 SE 026 764

AUTHOR Buchanan, Aaron D.
TITLE Estimation as an Essential Mathematical Skill. Professional Paper 39.
INSTITUTION Southwest Regional Laboratory for Educational Research and Development, Los Alamitos, Calif.
SPONS AGENCY National Inst. of Education (DHEW), Washington, D.C.
REPORT NO SWRL-PP-39
PUB DATE 31 Aug 78
CONTRACT NE-0-00-3-0064
NOTE 29p.

EDRS PRICE MF-$0.83 HC-$2.06 Plus Postage.
DESCRIPTORS *Basic Skills; *Calculation; Curriculum; Elementary Secondary Education; *Instruction; *Literature Reviews; *Mathematics Education; Measurement; *Research Reviews (Publications)
IDENTIFIERS *Estimation

ABSTRACT

Progress that has been made in teaching estimation is reviewed. The paper shows that although little has been done in research and development, there is a great deal of interest in developing estimation as a primary skill in school mathematics. It suggests that it is feasible to design instruction sequences in computation, measurement, and problem solving where estimation is developed at an early point so that it can become a tool for learning other skills and concepts. (MP)

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SWRL EDUCATIONAL RESEARCH AND DEVELOPMENT

Professional Paper 39

August 31, 1978

ESTIMATION AS AN ESSENTIAL MATHEMATICAL SKILL

Aaron D. Buchanan

ABSTRACT

Although estimation as a skill has long been held in the highest regard, there has been relatively little research and development that would make it a more workable part of the mathematics curriculum. Estimation is not only a valuable skill in its own right, it can also be a valuable pedagogical tool to be used in development of other important skills. By expanding the role of estimation in the curriculum, it is possible to give pupils more contact with mathematical structures involved in number systems, numeration, and arithmetic processes that are part of computation and measurement. In an expanded role, estimation should be developed earlier in the K-6 sequence so that it can be used at the very beginning of instruction on computation algorithms to provide pupils with greater capability to determine when the results of computation are sensible. This paper gives a new rationale for teaching estimation earlier and more comprehensively than it is taught now. Existing instruction has apparently had little impact on pupils' inclination and ability to do estimation. A better design for instruction that builds upon the idea of teaching estimation as an essential mathematical skill is needed. As part of the design work, there are strong implications for the role that must be filled by an integrated program of research and development in meeting the needs for new instruction that is systematic and effective.

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ESTIMATION AS AN ESSENTIAL MATHEMATICAL SKILL

Aaron D. Buchanan

In spite of its obvious and widely recognized potential for helping students and adults to obtain quick, sensible answers to mathematical problems, estimation continues to slip through the structural cracks of school mathematics. For over forty years, the literature of mathematics education has regularly registered a flow of good intentions regarding the topic, and recent recommendations for improving the mathematics curriculum have specifically emphasized estimation as one of a select group of essential mathematical skills (Bell, 1974; Edwards, Nichols, & Sharpe, 1972; National Advisory Committee on Mathematical Education, 1975; National Institute of Education, 1975; National Council of Supervisors of Mathematics, 1977). But the literature of mathematics education is not the school mathematics curriculum, and recommendations are not always translated into classroom practices.

Instruction in estimation is not something that has been tried and failed; it has not been tried on any sustained or systematic basis. There have been a few good essays on the kind of estimation activities that could become part of mathematical learning, but there have not been comprehensive strategies for integrating such activities into broad programs of school instruction.

The integration of estimation into the mathematics curriculum requires more than simple introduction of appropriate instructional activities followed by some direct practice. The objective is to develop a skill, not merely to provide experience. In order to develop estimation as a usable skill, there must be a long-range strategy for sustained use and application, particularly in the elementary school, that extends over several years of mathematics instruction. The investment in time cannot be negligible, but it is possible that this investment may have greater payoff than just teaching pupils how to estimate the results of computation. In the long run, estimation may be able to play a significant role in raising the general level of quantitative literacy and mathematical understanding among students and adults, an objective that has been fundamental to the aspirations of every generation of "curriculum reform."

The purpose of this essay is to outline a strategy for integrating estimation into the arithmetic curriculum, and to present some arguments for expanding the basic role of estimation in teaching about numbers, operations on numbers, measurement, and problem solving.

Background of Estimation in the Elementary School Curriculum

During 1975 and 1976, the Arithmetic Teacher featured a series of interpretative reports on mathematical competencies across the nation. The reports were based on data obtained in a 1972-73 mathematics survey conducted by the National Assessment of Educational Progress (NAEP) among students at ages 9, 13, and 17, and among young adults. One of these reports (Carpenter et al., 1976) was devoted entirely to estimation. This fact is important because, had the series of reports been organized around the major skills that are actually taught in the curriculum, estimation could hardly have merited a single paragraph. To quote from the report:

> There is general agreement that developing skill in estimation should be a major objective of the school mathematics curriculum . . . In fact, estimation is more important than precise calculation for many common uses of mathematics.
>
> Given the importance of estimation, it is perhaps one of the most neglected skills in the mathematics curriculum. (p. 296)

The results of this neglect are apparent a few paragraphs later where it is reported that 9- and 13-year-olds showed little if any skill in performing simple estimates of the results of computation. Seventeen-year-olds and young adults were better, but close to half of them still were unable to do two or three different kinds of estimation problems.

Estimation in the Curriculum

Although Carpenter et al. (1976) only infer that there is little instruction on estimation in most mathematics programs, the inference is easy to verify. A survey* of six programs for elementary school mathematics shows that the average number of lessons which have very much to do with estimation directly is low:

Grade	Average Number of Lessons
3	Less than 1
4	3
5	3
6	3

*Conducted by Ms. Jannine Perkins at SWRL

Obviously, estimation does not have a high priority in current mathematics programs. In the past, programs have spent a little more time on estimation, but their treatment of this area has often been haphazard. Flournoy (1957) surveyed activities on estimation in several fifth grade programs and found that the number of pages of instruction varied from nine to forty-three in an average textbook having 300 to 350 pages. Most often, textbook activities were aimed at finding sensible answers. Across several programs, Faulk (1962) found exercises on estimating the results of computation on whole numbers, estimating measurements and averages, and estimating answers to verbal problems, but only some of these exercises were in any single program.

The reasons for neglect of estimation in school programs are varied. Certainly, this is not an instance where other skills are simply more important. More likely, many of them are simply easier to teach. In part, a better rationale for teaching estimation is needed, but is only now beginning to emerge.

Over the past 30 years, the arguments for teaching estimation have frequently been indirect. Prior to 1960, there was some interest expressed in improving the teaching of mathematical approximation and approximate computation (Bakst, 1937; Harper, 1941; West & Shuster, 1943; Gager, 1957; Shuster, 1957). These areas commonly included such topics as significant digits, rules for computation involving significant digits, precision and relative error in measurement, and specific methods for obtaining mathematical approximations of numbers such as π and $\sqrt{2}$. While a stronger case could have been made for developing estimation as a form of approximation involving rounded numbers and mental judgments about measurement, it was seldom given more than a passing reference.

More often, estimation has been associated with the teaching of mental arithmetic as a necessary complement to pencil-and-paper algorithms for doing exact computation. Between 1930 and 1960, there was some concern that pupils were becoming too dependent on pencil-and-paper algorithms and that more practice on mental arithmetic was needed. Mental arithmetic had been popular prior to 1900, but was used as a form of exercise to improve general mental discipline. The arithmetic skills, themselves, were only vehicles for inducing mental activity which often involved intense oral drill on computation problems requiring several steps to complete. After 1900, the theory of mental discipline fell into such disrepute that it was difficult to maintain a place in the curriculum for any form of mental arithmetic. Efforts to revive mental arithmetic began about 1930 and reached their peak in the decade just prior to the "new" mathematics (Jordan, 1929; Hall, 1942, 1947; Moser, 1953; Sauble, 1955; Petty, 1956; Wandt & Brown, 1957; Flournoy, 1957, 1959). The rationale for the revival was that mental arithmetic actually was used more in activities outside the classroom than was pencil-and-paper computation. Therefore, pupils should be involved in straightforward kinds of computation embedded in problems that closely resembled experiences they might encounter in daily life. Estimation as a form of approximation was frequently included as one of the socially

useful skills that pupils needed to develop, but most often mental arithmetic focused on exact computation without the aid of pencil and paper.

Since 1960, there has been less interest in teaching mental arithmetic, probably due to the emphasis placed on learning mathematical structure in the new programs that have been developed. There has also been a shift in the philosophic orientation of school mathematics away from drill and practice to discovery learning and independent inquiry which are not compatible with any serious attempt to develop mental arithmetic as a useable skill.

During the period since 1960, estimation has maintained its own identity as part of the curriculum aimed at teaching students how to tell when answers to computation and measurement problems are sensible (Sauble, 1955; Brown, 1957; Schmid, 1967; Trimble, 1973; Holmes, 1975; Easterday, 1978). The objective of instruction has been to develop strategies for obtaining approximate answers rather than exact ones, so the need for oral drill and practice does not seem to be as overpowering as it was with mental arithmetic. As a consequence, it has been possible to think about estimation as a part of learning about mathematical structures, since rounding and approximation do require some basic understanding of the numeration system and fundamental arithmetic processes, but little has been done so far in research and development that would improve instruction.

Research on Estimation

Very few research studies have been reported that focus directly on estimation. In addition to the report by Carpenter et al. (1976) on the results of the National Assessment of Education Progress, Fault (1962) and Corle (1963) reported on surveys they conducted on how well pupils could estimate. Both found that at the elementary school level, estimation skills were weak. Dickey (1934), Nelson (1967) and Ibe (1971) reported on the effectiveness of instruction in improving pupils' ability to estimate. Dickey's study, showed no special gains in overall achievement in problem solving among pupils who spent part of their time practicing estimation over pupils who only had practice on problem solving. Nelson (1967) and Ibe (1971) reported that instruction on estimation improved pupils' performances on estimation tasks and appeared to improve understanding of certain mathematical concepts.

Paull (1971) studied eleventh graders' ability to do different kinds of estimation in both computation and measurement. He found that these students were not consistent in their abilities to estimate length, area, and answers to numerical computation. The correlation between scores on estimates of length and estimates of numerical computation, for example, was quite low. In addition, he found that the ability to estimate the results of computation did correlate

positively with the ability to solve mathematical problems by trial and error. These results are tenuous, given other data which show the tepid performance of pupils on most kinds of estimation, even in the eleventh grade.

The reasons for so little research on estimation are not clear. Carpenter et al. (1976) suggest that estimation as a process is difficult to measure and that criteria for good estimates are ambiguous. However, there is a great deal of research available in other areas of mathematics (e.g., problem solving) where abilities are also hard to measure. One of the major difficulties in measuring pupils' ability to estimate is that their performance is invariably confounded with their ability to do computation. But for pupils who have not yet learned to compute directly, it should be relatively straightforward to find out how well they can learn to estimate. Whether they will continue to estimate after they have learned how to do direct computation is another question.

A Broader Role for Estimation

It is possible that the case made for teaching estimation over the past 30 years has not proposed benefits that apply broadly enough across the mathematics curriculum as a whole. The most commonly perceived benefit has been that estimation would serve as a form of quality control over computation. In this sense, estimation has been viewed in a subordinating role. Pupils would be taught to compute first, and then they would be taught to estimate so they could tell if their answers were sensible.

There are indications that more important benefits can be anticipated from systematic instruction on estimation, and that estimation need not be subordinate in the structure of the mathematics curriculum to exact computation. Two important essays illustrate some of the possibilities. Both essays (Payne & Seber, 1959; Bright, 1975) were written for the Yearbook of the National Council of Teachers of Mathematics. Although fifteen years apart, they are significant building blocks in the rationale for assigning estimation a broader role in the curriculum for the elementary school.

The first part of the Payne and Seber essay on "Measurement and Approximation," is really the most relevant to elementary school mathematics. Here, the authors give a series of recommendations and useful suggestions for teaching pupils how to estimate the results of numerical computation, how to make estimates of the number of objects in a large aggregation, and how to make estimates in measurement. More important, however, is the way Payne and Seber describe estimation within the structure of mathematics. Previously, estimation was thought of as little more than a technique for computing with rounded numbers as a check on "real" computation. Payne and Seber show how estimation is related to the concept of "more than," "less than," and "between," and

how the idea of improving estimates is the basis for the entire area of mathematical approximation. According to the authors:

> Implicit in all work on estimation is the attempt to get pupils to think about the result and make an estimate before plunging directly into exact paper-and-pencil computation. The major attempt is to get pupils to deal with numbers sensibly. (p. 189)

The students' ability to deal with numbers sensibly would benefit from more experience with the concepts of "greater than," "less than," and "between," and from a greater understanding of characteristics implicit in the number system.

The underlying relationship between estimation and the structure of mathematics, as Payne and Seber expressed it in 1959, was more one-sided than it might be if it were written now. At that time, it appeared estimation would benefit from more experience with mathematical structure. What the authors did not express (but certainly must have recognized) is that more experience with estimation could be expected to improve students' understanding of the structure of the number system, itself, by increasing the level of contact pupils would have with certain aspects of this structure. The point is an important one, and to understand it requires a quick review of what was happening in mathematics education at the time the Payne and Seber essay was written.

The 1959 Yearbook was one of the first responses of the National Council of Teachers of Mathematics to the reform that was taking place in mathematics instruction at that time. The curriculum was being reworked to place more emphasis on elements of mathematical structure and, at the secondary level, these elements seemed to be important in their own right. At the elementary level, a little different rationale was taking shape. It was clear that many people in research and development wanted to accomplish at the elementary level what was already taking place in secondary curriculum: organize more of the content around mathematical structure. The rationale that was given was more practical; a better understanding of underlying mathematical concepts was to make it easier to teach pupils how to compute, and, by the same argument, how to estimate.

In retrospect, structure was and is intrinsically important in the elementary curriculum just as it is at the secondary level. Estimation could not only benefit from ideas embedded in the structure, it should be a basic part of learning the structure. Experience is at the heart of dealing with numbers sensibly, and estimation should play a significant role in increasing this experience.

By 1975, when Bright wrote his essay for the Yearbook, mathematics education had the benefit of 15 years of experience with reform in the curriculum. Significantly, the title of the essay was "Estimation as Part of Learning to Measure." In many respects, Bright's essay took

Payne and Seber's preliminary work and carried it forward an important step. While estimation of measurements may be intrinsically important for a variety of practical reasons, it could also be an important part of learning some elements of mathematical structure, in this case, the elements involved in the measurement process.

There has long been a kind of kinship between estimation and measurement since both processes deal with the notion of approximation. Payne and Seber, in 1959, had shown how the idea of mathematical approximation grew out of these two processes. During the period after 1959, measurement as a process received a growing amount of attention from research and development. The objective was to place considerably more emphasis on the relationship between measurement of an object and the size of the unit. Pupils were to understand that measurement units are somewhat arbitrary and that relationships between two measures are based on the size of the units involved. Bright's rationale was that estimation occurring early in the sequence for teaching measurement would raise the level of appreciation and understanding of the role of the unit. There was no intent to play down the practical importance of being able to estimate in measurement, which had been the basis for one of Payne and Seber's arguments. Instead, the objective was to broaden the role of estimation in the overall strategy for providing learning experiences in mathematics.

Bright did not go into much detail as to why more and earlier experiences with estimation would improve pupils' understanding of the measurement process, but three points can be extrapolated from his rationale:

(1) Estimation can increase the density of pupil experience with units and measurements in the sense that more contacts with measurement problems are possible within a fixed amount of instructional time. This does not mean that experiences involving actual measurement of objects is not important. Obviously, it is essential. The point is that by teaching pupils how to estimate in measurement, it is possible to provide a mix of activities in estimation and direct measurement that will increase the contact pupils have with both the structural aspects of the measurement process and its practical applications.

(2) Estimation provides a means to broaden experience with measurement. For some dimensions of the physical world, there is no practical way for pupils to apply the measurement process except through estimation (e.g., weight of a football stadium, great distances, number of pores in the skin, and so on).

(3) Improving estimates in measurement increases pupil contact with relationships between units and measures, relationships that are implicit in the measurement process. If pupils consistently overestimate the length of different objects, for example, then their image of the unit as part of an object, may be too small.

Estimation in Numerical Computation

Estimating the results of numerical computation has been treated as a passive skill for ensuring correct answers to computation problems much like the "checking" procedures which are taught inconsistently in the elementary curriculum. The technique is usually to instruct pupils to use some kind of approximate procedure in advance of direct computation in order to have a check on the answer. Most of pupil time is spent learning how to do direct computation, and pupils generally come to regard estimation as an annoyance which they can ignore. What they have learned is to place their confidence in the step-by-step procedures they know for doing computation. If more quality control is needed, then it seems likely that pupils simply try to be more careful so as not to make so many mistakes.

The frustrations that arise in trying to teach estimation can be traced directly to the structure of the curriculum in elementary school mathematics. The defacto purpose of this instruction is to deliver skills in computation, which most often means skill in manipulating the steps in computation algorithms. Other skill areas such as problem solving are considered to be highly important, but computation algorithms receive the most systematic practice in instruction.

The parts of the elementary curriculum that deal with numbers, relationships between numbers, operations on numbers independent of the complex algorithms for doing computation, and information processing or problem solving, have consistently received less emphasis in instruction than pupils would find useful. The concern over developing an intuition about numbers is rooted in the realization that too little practice is provided in noncomputational areas of instructional programs. To be sure, some practice is provided on skills other than computation, but it is often sequenced badly. Aside from being able to count the objects in small aggregations and to "recognize" larger numbers, the objectives for developing an intuition about numbers are not clear. In the area of "recognition," for example, the objective is often to know enough about place value to be able to translate number names back and forth from standard, numerical form to word form.

More experience with numbers, relationships between numbers, and operations needs to be built into the structure of the curriculum prior to, or at least independent of, instruction on formal algorithms for computation. At present, there is no good mechanism for considering operations on numbers with more than two digits independent of the algorithms for computation. Any understanding of the order relationship between numbers and how this relationship is affected by different arithmetic operations is either confined to one-digit and small two-digit whole numbers, or it must be confounded with an ability to compute. When skills are confounded with development of computation skill, they must often be delayed until near the end of

elementary school. In most instances, it takes two to three years for pupils to develop one of the major computation algorithms. For example, the algorithms for multiplication and division of whole numbers are not seriously introduced until late in grade 4, and for many pupils, they are still not developed into anything useful by the end of the K-6 instructional sequence.

Estimation provides an avenue for significantly increasing pupils' experience with operations on whole numbers and fractions. There is every reason to believe that pupils can learn simple methods for approximate computation almost as soon as they learn computation facts, that this skill will provide the technical capability for estimation, and that the amount of time required to learn how to estimate will be modest in relation to the payoff.

Until now, there has been little attempt to teach estimation while pupils are still naive about formal algorithms for doing exact computation. Yet, early instruction in estimation, before pupils get involved in trying to learn how to perform formal algorithms, could avoid some of the previous difficulties that surrounded attempts to teach estimation skills. First, estimation becomes a way of doing problems rather than checking out results, and, in this sense, it becomes an active process. Second, it is a quick method for increasing pupils' technical capabilities for doing arithmetic early in the elementary years, and these capabilities can be readily applied to such areas as measurement and problem solving.

Compared to the development of formal algorithms, the abbreviated process involved in estimation should be relatively easy to teach. To teach some estimation as early as grades 2 or 3, for addition and subtraction of whole numbers, and grades 3 or 4, for multiplication and division with whole numbers poses no real, technical difficulties. In situations where all of the computation facts have not been thoroughly mastered, there are pedagogical techniques for using selected facts to develop new skills and technical capabilities.

Potential Benefits of Estimation Instruction

The payoff for early instruction in estimation followed by regular application and reinforcement would appear to have significant impact on four areas of mathematics instruction where the need for improvement in present practices is widely recognized.

Estimating to determine whether answers are sensible. Obviously, pupils will have plenty of contact with the technical aspects of doing estimation because they are naive about formal algorithms for doing exact computation. The idea is to invest a modest amount of time at about the point where computation facts are first being practiced seriously and quickly expand the level of pupils' technical capability

for operating on numbers. A significant amount of practice would be provided in computing with rounded numbers some months in advance of the introduction to formal algorithms for exact computation. By having the technical capability for doing estimation in place before instruction on formal algorithms begins, estimation can be used as an important resource throughout the entire two- to three-year sequence for learning how to compute. It is important, however, that technical skills for doing estimation be developed, practiced, and consolidated early, not just a few lessons before computation with formal algorithms is begun. In this way, we can extend the meaning of arithmetic operations independent of the complex algorithms for doing computation. Direct computation should be regarded by pupils as one method for obtaining exact answers, along with hand-held calculators, but it should not be synonymous with the concept of operation on numbers, as it is now.

As a process for determining whether answers are sensible, estimation would have the same kinds of application as exact computation. When pupils are learning how to process information and solve problems, and when they are using computation with measurement, metric geometry and conversion of units across measurement systems, estimation would be available for use as a check on direct computation.

Estimating as part of learning about mathematical structure. Learning how to estimate and learning about mathematical structure should be closely integrated in the K-6 instructional sequence, and they should prove to be mutually reinforcing. In this area, skills develop in cycles, and it would not be very meaningful to try to identify rigid prerequisites, either for learning how to estimate or for learning about certain relationships that are a basic part of the structure of mathematics.

Initially, estimation requires a rich background in counting with small numbers, extensive experience with the concepts of "more," "less," "equals," "between," and "nearer," as applied to small numbers, along with mastery of some computation facts (at least the easy ones). Estimation involves regular and varied kinds of practice with logical relationships like the following:

5 + 5 equals 10

8 is more than 5

So 5 + 8 is more than 10.

This relationship is a form of transitivity which cognitive research has generally shown to be difficult for young children (Smedslund, 1963; Brainerd, 1973, 1974; Hooper and Toniolo, 1977) partly because they have not had much direct experience. It is powerful because:

(1) it taps into one of the most basic building blocks in mathematics, relations between numbers, and

(2) it is the foundation for other activities that come later in the curriculum, most of which are part of acquiring a "sense" about numbers.

At a higher level, estimation combines recall of basic facts with rounding of whole numbers to multiples of 10 or 100 into a technical skill for computing with rounded numbers. An important relation at this stage is "about":

346 is about 300

562 is about 600 (or 500)

Extending this relation to addition:

346 + 562 is about ___?___

At this point, the process can be tied back to mathematical relationships that have been learned earlier. For example, until now, there have not been any very precise rules in this sequence for rounding. Better results can be obtained by experience with the order relationships that were begun earlier.

346 is between 300 and 400,

but it's closer to 300

The emphasis is on the concepts of "between" and "closer" or "nearer" and how these concepts can be used to get better estimates. Does an exact answer to 346 + 562 have a purpose here? Probably. It is, after all, the best estimate. How do pupils obtain exact answers without a formal algorithm? It appears that the hand-held calculator has tremendous potential in instances like this one where pupils are learning to distinguish between estimates and exact answers and are getting ready to explore techniques for obtaining better estimates.

The idea of improving estimates involves more experiences with the order relation as a key element of mathematical structure.

346 is between 300 and 400

562 is between 500 and 600

So 346 + 562 is between 800 and 1000.

The estimation process can be improved by tightening the interval using numbers such as 300 and 350, but regular instruction does not necessarily have to go in this direction. Increasing the precision of estimates beyond a certain point can be an extension topic in actual classroom instruction depending on how comfortable pupils are with the

basic processes involved in estimation and how well the teacher has developed a basic motivation for improving the results of estimation. The improvement of estimates could be used with some pupils as a motive for introducing formal algorithms for computation. For example:

346 is between 340 and 350

562 is between 560 and 570

To obtain the interval for 346 + 562, it is necessary to learn how to add with hundreds and tens and to do some regrouping. If this introduction involves too great a leap, then 346 + 522 can be used and no regrouping is required.

346 is between 340 and 350

522 is between 520 and 530

Using estimation as one vehicle for integrating the hand-held calculator as an instructional resource. Close relationships can be established between the development of estimation as a mathematical skill and use of the hand-held calculator as an important resource for mathematics instruction. In a sense, they are both resources to be used for placing pupils in contact with certain elements of mathematical structure and for development of other important skills in the mathematics curriculum. It has been suggested earlier that by using estimation as a tool for obtaining approximate answers and the hand-held calculator as a tool for obtaining exact answers, pupils can be given a broader experience in the relationships between numbers and the operations of arithmetic, and it can be done independent of complex algorithms for doing computation. The important thing to note here is that we have two relatively efficient methods for interacting with mathematical operations and that instruction can play one method against the other to provide pupils with a greater range of intuition-building experience with numbers. In addition, estimation becomes a check on the reasonableness of answers obtained by using the calculator, while the calculator becomes a source of exact answers in initial instruction on estimation and in developing techniques for making estimates more precise. It is most important to recognize that estimation improves the prospects for utilizing the potential of the hand-held calculator in mathematics learning. It does not merely give respectability to a piece of technology that might not otherwise be needed for teaching. Either the calculator or estimation could be used separately to improve the quality and quantity of pupils' experiences, but an integration of the two resources opens up a range of experiences that neither resource could approach alone. In any case, these resources expand the possibilities for teaching mathematical skills; they are not just different ways of teaching skills that are already a working part of the elementary school curriculum.

Estimation as an essential tool in teaching problem solving in arithmetic. Estimation as a skill increases the flexibility of the mathematics curriculum for teaching pupils how to process information and to solve problems. Basically, this area of mathematics instruction refers to pupils' work with graphs, charts, and verbal problems, but more specifically, it refers here to interpretation of quantitative information embedded in a variety of contexts including: pictures, physical models, spoken and written communications, displays, signs and charts, experiences outside the classroom including jobs, consumer problems, government activities and requirements, social issues, sports and recreation, and other channels or special areas of communication.

Until now, most instruction in these areas has been subordinated to emerging skills for exact computation. Instruction in processing information either had to wait until computation skills were in place, which for many pupils occurred very late in elementary schools, or pupils were faced with trying to use computation skills in learning how to process information when they didn't yet know how to compute. The result has been disappointing. Pupils have often been forced to try to learn two major kinds of skills simultaneously. Little progress has been made in problem solving because pupils could not get past their concern over computation. The problem is widely recognized, and estimation is one way to deal with it. There is, or should be, little concern initially with obtaining exact results on problem solving. Estimated results serve the same purpose, since the concern is for learning how to set up solutions to problems that are posed and how to answer questions that are asked. Ideally, instruction in information processing and problem solving should be as free from pupil concern over actual computation as possible.

This aspect of estimation instruction permits skills in information processing to be developed concurrently with skills in reading. In addition, the skills in reading are actually enhanced by the additional experience provided in specific content areas and areas of application. To some, it might seem that objectives for teaching information processing and problem solving could be accomplished solely through use of the hand-held calculator, with little if any need for skill in estimation; but this is an over-simplification of objectives in mathematics instruction and fails to take into account the slowness with which new technology is assimilated into classroom practice. When the calculator is available, estimation assumes its historic role of enabling pupils to check on the results of exact computation. Even now, there is some concern that pupils who use calculators in problem solving often engage in "button pushing" with little sense for the significance of results. When numbers are entered incorrectly or buttons are pushed in error, estimation serves as a basic form of quality control.

Summary. Aside from being quite feasible, the early development of estimation skills yields significant advantages over current practice:

(1) It provides pupils with a capability for determing whether answers to numerical computation are sensible.

(2) It allows pupils to develop a greater sense for numbers and how they behave in operations of arithmetic before so much close attention must be paid to the step-by-step procedures involved in computation.

(3) It complements and expands the potential of hand-held calculators as tools for teaching mathematics.

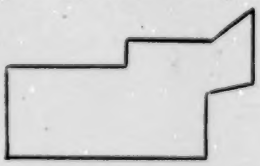
(4) It provides for more flexibility in timing and development of skills in information processing and problem solving which have always been delayed because of the excessive amount of time required to acquire proficiency with computation skills.

Estimation in Measurement

A good model for teaching estimation in conjunction with measurement has been outlined by Bright (1975) in his essay on "Estimation as Part of Learning to Measure." The objective is to increase pupils' experience with the relationship that exists between the unit of measurement and the actual measure that is obtained. When asked to estimate the height of a chair, for example, the learner must have a mental image of some unit of length, say a decimeter, and must somehow "see" the number of those units placed end to end that would be as tall as the chair. The "seeing" is a skill that obviously develops in direct proportion to the amount of experience the learner is given. Some technical skills can be acquired for improving the process. For example, the height of a chair can be compared mentally with the known height of a familiar object, such as the height of the person doing the measuring. However, there is no real substitute for direct experience.

Estimation plays an essential role in providing more contact between the learner and mathematical relationships that are implicit in the measurement process. In providing pupils with experience in estimation, there is a need for variety as well as a need for a great deal of repetition. For example, professional foresters often become highly proficient at estimating the number of "logs" in a standing tree. A log, in this case, is a standard unit of 16 feet. The activity involves little more than visual and mental imagery and a great deal of experience. It is not a case where the height of the tree is estimated in feet and then divided by 16. There is little research to show how well this kind of skill transfers to other kinds of estimation, such as judging distance, that have not been practiced directly, but it would seem to be a safe bet that the amount of transfer is small. How well abilities transfer across even closely related kinds of measurement isn't known, but it appears that a vareity of practice is needed.

What is acquired through practice in estimation? Bright (1976) suggests that pupils acquire a sort of heuristic for combining enough copies of a unit to "fit" the object being measured. For example, in estimating the area of an object such as the one shown below, pupils learn to rearrange the irregular parts mentally until



there are enough to fill a unit. In estimating the volume of a garbage can, pupils learn to imagine the units stacked mentally so that as little "empty" space is left as possible.

Pupils also learn to adjust the estimation process using feedback from exact measurements. If estimates are consistently too high, the pupil can:

(1) simply reduce the size of the final estimate by a small amount,

(2) try to stretch his image of the unit that is involved,

(3) try to improve his image of the combination of units together to fill the dimension of the object.

In practice, different kinds of estimation involve different processes. Estimation of length or height mostly involves mental imagery. In estimating area and volume, pupils often guess the linear dimension of an object and then apply a standard formula. For example, in estimating the area of a rectangle, it is likely that pupils might guess the length and width and then use the guesses in a formula for area: $A = L \times W$.

A strategy for providing pupils with experience in estimation might include the following steps.

(1) Providing a small amount of experience with a unit in direct measurement. This serves as an introduction to the measurement process and is repeated each time a new unit (meter, centimeter, foot, . . .) or a new dimension (length, mass or weight, area, . . .) is introduced.

(2) Establishing a rough interval that safely "traps" the actual measure. The size of the interval can be quite large, at least during initial instruction. This kind of activity is parallel to order-of-magnitude estimates in computation. For example, one might expect pupils to determine that the length of a car would be somewhere between one meter and one hundred meters or, later, that it would be between one meter and ten meters.

(3) Making an estimate of the actual measure. Frequently, at least at the beginning of instruction with a new unit or dimension, pupils would verify their estimates by actual measurement, but not always.

(4) Expanding the amount and variety of experience with a single kind of measurement (e.g., length). Pupils would have experience with various ways of expressing length such as "how long," "how high," or "how far."

(5) Comparing estimates with familiar objects whose measures are known. Such objects frequently include parts of the body (the width of a finger is about a centimeter), the

height of a table (about one meter), the temperature of water from the faucet (about 15 degree centigrade).

(6) Developing techniques for improving estimates.

(7) Estimating with different units. (For example, using both a centimeter and a foot to estimate the height of a chair.)

(8) Estimating where a unit is specified. (Estimate the height of a room in meters.)

(9) Estimating where a unit is not specified. (Estimate the height of the room.)

(10) Estimating lengths, distances, and other quantities represented in pictures, models, scale drawings, etc. Here, it is likely that some sort of scale is given especially during the early part of instruction. However, pupils might later draw inferences about the scale from familiar features in the picture or model.

(11) Estimating dimensions where it is hard to measure directly and to make mental judgments about measures (e.g., weight or mass, the volume of certain kinds of containers, speed, temperature, long distances.)

(12) Estimating dimensions that cannot be measured directly (e.g., weight of a stadium, height of a tall building, area of a small lake or pond.)

(13) Estimates which are not based directly on units. This kind of estimate involves direct comparison with objects whose measures are known, but it does not involve the mental combination of units to fit a space or object. For example, the height of a classroom is about three meters; the teacher's height reaches a little more than halfway to the ceiling. Therefore, his height would be a little more than a meter and a half.

The steps outlined above are not intended to be a complete sequence for teaching estimation, nor do they need to be followed in instruction in the precise order given. Generally, the flow of activities should be in about the order shown. Many steps would overlap, particularly when different kinds of measurement were being introduced. Some steps form instructional cycles that would be repeated as new measurement concepts are being taught.

Estimation of Quantity

Estimating the number of objects in a large aggregation is part of the model just described for estimation in measurement. This form of estimation typically involves a kind of sampling process where a small part of the aggregation is separated, the objects in that part are counted, and the part is treated as a unit of measurement. By estimating how many parts of this size are involved in the total aggregation and then multiplying by the number of objects in the part, one can obtain an estimate of the total number of objects.

Estimation of quantity is closely related to some of the elementary notions in statistics, since both involve a process of sampling. In an instructional sequence, the sampling process involved in estimating quantities could be used as a way to lead into simple concepts and processes involved in statistics. Instead of looking just at the total number of objects in a sample, instruction can begin to focus on the fraction or proportion of objects that have a certain property and use this fraction to estimate the total number of objects that have that property.

Summary

This essay has reviewed progress that has been made until now in teaching estimation. While little has been done in research and actual development, there is still a great deal of interest in developing estimation as a primary skill in school mathematics. Expanding the role of estimation in the curriculum can be based on using this skill to learn more about mathematical structures in systems of numbers and numeration and in arithmetic processes involved in computation and measurement. It appears to be highly feasible to design instruction sequences in computation, measurement, and problem solving where estimation is developed at an early point on the instructional time line so that it can become a tool for learning other skills and concepts.

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